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# A NOTE ON THE NONLINEAR RESPONSE OF AN ELASTIC BEAM ON A FOUNDATION TO A MOVING LOAD

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**Abstract**—The nonlinear response of an elastic beam to a moving transverse load is studied, using a special perturbation method. Solutions are obtained that remain valid throughout some neighborhood of the critical speed of the linear beam theory. It is found that in general, depending upon the type of dominant nonlinearity in the beam, either the subcritical response or the supercritical response may be continued up to the critical speed and even beyond. The solutions also show how the transitions from a subcritical response to a supercritical response and vice versa take place near the critical speed.

## NOTATION

| | |
|---|---|
| $\lambda$ | $[EA/(4k_1r^4)]^{\frac{1}{4}}$ |
| $\xi$ | $x/(\sqrt{(2)}\lambda r)$, dimensionless distance in moving coordinate system along undeformed beam neutral axis |
| $\eta$ | $y/(\sqrt{(2)}\lambda r)$, dimensionless transverse displacement |
| $\gamma$ | $u/(\sqrt{(2)}\lambda r)$, dimensionless axial displacement |
| $v$ | $\lambda V(\rho/E)^{\frac{1}{2}}$, dimensionless load velocity |
| $\alpha$ | $2\lambda^2r^2k_2/k_1$, dimensionless modulus for material nonlinearity of foundation |
| $\beta$ | $k_3/(2k_1)$, dimensionless horizontal modulus for foundation material |
| $\Delta$ | $2\lambda^2N/(EA)$, dimensionless load intensity |
| $\mu$ | $E_1I_4/(2\lambda^2r^2EI_2)$, dimensionless modulus for material nonlinearity of beam |

In the above expressions, $x$, $y$, $u$, $E$, $E_1$, $I_2$, $I_4$, $A$, $r$, $k_1$, $k_2$, $k_3$, $\rho$ and $N$ have the following meanings, respectively:

| | |
|---|---|
| $x$ | $X-Vt$, distance in moving coordinate system along undeformed beam neutral axis, in which $X$ = distance in fixed coordinate system along undeformed beam neutral axis, $V$ = load velocity and $t$ = time |
| $y$ | transverse displacement of beam |
| $u$ | axial displacement |
| $E, E_1$ | elastic constants that relate stress and strain in a beam: $\sigma = Ee-E_1e^3$, in which $\sigma$ = stress, $e$ = strain and $E_1 \geq 0$, $E_1/E = O(10^3)$ if $E_1 \neq 0$ |
| $I_2, I_4$ | moment of inertia and fourth moment of beam cross-section |
| $A$ | area of beam cross section |
| $r$ | $(I_2/A)^{\frac{1}{2}}$ |

| | |
|---|---|
| $k_1, k_2, k_3$ | foundation stiffness factors that relate restoring force and displacement: $F_y = k_1 v + k_2 v^3$ in which $F_y$ = vertical restoring force/unit length, $k_1 > 0$ and $k_2 \geq 0$, $k_2/k_1 = O(10^3)$ if $k_2 \neq 0$; $F_x = k_3 u$ in which $F_x$ = horizontal restoring force/unit length, $k_3 > 0$ and $k_3/k_1 = O(1)$ |
| $\rho$ | density of beam material |
| $N$ | transverse force |

## INTRODUCTION

THIS note concerns the response of an elastic beam, of infinite length and supported on an elastic foundation, to a steadily moving transverse load. We shall consider here only those loads with speeds near or equal to the so-called "critical" speed of the linear beam theory. It is known (Kenney [1], Steele [2]) that solutions of the present problem on the basis of the linear theory are inadequate because of the occurrence of large-amplitude steady state responses, or "resonances", even if the load intensity is kept small. In [2], Steele has treated the present problem using the nonlinear beam theory. He obtained four distinct perturbation solutions—one a Poincare type expansion that is valid when the load speed is strictly subcritical, one a Lindstedt type expansion that is valid when the load speed is strictly supercritical, and two solutions for loads moving exactly at the critical speed, one of which being valid when the geometrical nonlinearity is predominant and the other when the material nonlinearity is predominant. It may be pointed out that for the success of the perturbation method used by Steele one has to guess at the correct forms of the perturbation series. Furthermore, as the ranges of load speeds for which Steele's solutions are valid do not overlap, it remains obscure as to how his solutions may be related to one another.

For moving load problems for continuous systems such as the beam problem considered here, it appears highly desirable to obtain nonlinear solutions which are valid throughout some particular neighborhood of a critical speed. Such solutions not only will reveal the response of a system to a load moving near or at a critical speed, but also will provide information on how the transitions from a subcritical response to a supercritical response and vice versa take place near the critical speed. To this end, a special perturbation method was recently developed by the present authors and applied successfully to the problem of a nonlinear elastic string [3].

The purpose of this note is to apply the perturbation method developed in [3] to the present beam problem and to obtain, in a systematic manner, nonlinear solutions that are valid in the neighborhood of the critical speed. An essential feature of this special perturbation method consists in expanding both the solution and the load speed, as well as one or more parameters in the problem if necessary, as power series in a small perturbation parameter. The dependence of the load speed and the other parameters upon the perturbation parameter is then so adjusted that meaningful perturbation solutions are obtainable. It is through the dependence of the load speed upon the perturbation parameter that one may, by starting with an unperturbed linear solution involving a subcritical or supercritical speed, construct nonlinear solutions which are valid throughout some neighborhood of load speeds containing the critical speed.

We find that in the neighborhood of the critical speed there exist essentially two distinct nonlinear solutions. One of these is the continuation of the subcritical solution and the other that of the supercritical solution. Depending on whether the geometrical or the material nonlinearity is predominant, one or the other of the two solutions may be continued up to the critical speed and even beyond.

## GOVERNING EQUATIONS AND LINEAR SOLUTIONS

The same set of nonlinear equations for the steady-state beam motions under a moving load as those derived by Steele [2] will be used. In dimensionless form the equations are

$$\eta'''' + 2v^2\eta'' + \eta = \mu[(\eta'')^3]'' + \alpha\eta^3 + 2\lambda^2\{\eta'[\gamma' + \tfrac{1}{2}(\eta')^2]\}' + \Delta\delta(\xi) + O(\eta^5), \tag{1a}$$

$$(\lambda^2 - v^2)\gamma'' - \beta\gamma = -\tfrac{1}{2}\lambda^2[(\eta')^2]' + O(\eta^4). \tag{1b}$$

$\eta$ and $\gamma$ here are, respectively, the transverse and axial displacements, and primes denote differentiations with respect to the moving coordinate $\xi$. The derivation of these equations may be found in [2], with the further simplification that, as we assume $\lambda^2$, $\alpha$ and $\mu$ to be large in comparison with unity (unless they vanish), nonlinear terms in the displacements which are not multiplied by such quantities have been omitted from equations (1). We remark also that, unlike Steele, we consider here the full beam ($-\infty < \xi < \infty$) with the concentrated load $\Delta\delta(\xi)$ term being added to equation (1a).

To complete the formulation of the problem we add the conditions of continuity of $\eta$ and $\gamma$ as functions of $\xi$ and of the boundedness of these solutions as $\xi$ tends to $\pm\infty$. Further physical considerations will also be given in case the above conditions do not determine the solutions uniquely.

The linearized form of equations (1) is well known. For small $\eta$ it follows easily from equation (1b) that $\gamma$ will be of the order $O(\eta^2)$, and thus may be omitted from consideration. Equation (1a) then reduces to

$$\eta'''' + 2v^2\eta'' + \eta = \Delta\delta(\xi). \tag{2}$$

For $v < 1$ the solution of equation (2) is

$$\eta(\xi) = \frac{\Delta}{2(1-v^4)^{\frac{1}{2}}}\mathrm{Re}\{\exp[i(|\xi|e^{i\Phi} - \Phi)]\}, \tag{3}$$

where

$$\Phi = \cos^{-1}\left(\frac{1+v^2}{2}\right)^{\frac{1}{2}}, \qquad (0 \le \Phi \le \pi/4) \tag{4}$$

and for $v > 1$, the solution of equation (2) is

$$\eta(\xi) = -\frac{\Delta}{2(v^4-1)^{\frac{1}{2}}}\left[\frac{1}{\Omega_a^{\frac{1}{2}}}\sin\Omega_a^{\frac{1}{2}}\xi \,.\, H(-\xi) + \frac{1}{\Omega_b^{\frac{1}{2}}}\sin\Omega_b^{\frac{1}{2}}\xi \,.\, H(\xi)\right] \tag{5}$$

where

$$\Omega_a = v^2 - (v^4-1)^{\frac{1}{2}}, \qquad \Omega_b = v^2 + (v^4-1)^{\frac{1}{2}}. \tag{6}$$

Neither (3) nor (5), however, is valid as $v$ tends to unity. In fact, when $v = 1$, equation (2) possesses no solution that remains bounded in $-\infty < \xi < \infty$. The speed $v = 1$, as we have normalized here, is the so-called "critical" speed. It may be pointed out that the linearized form of equation (1b) gives rise to another critical speed $v = \lambda$, which is assumed large and will not be considered here.

## NONLINEAR PERTURBATION SOLUTIONS

We now construct solutions of the nonlinear equations (1), using a modified perturbation method. We take the load intensity as our perturbation parameter and set

$$\Delta = \varepsilon. \tag{7}$$

We then expand $\eta(\xi, \varepsilon)$ and $\gamma(\xi, \varepsilon)$ as power series in $\varepsilon$. It is expected that both $\eta$ and $\gamma$ vanish when $\varepsilon = 0$. Furthermore, equation (1b) shows that $\gamma$ is of the order $O(\eta^2)$. We thus write

$$\eta(\xi, \varepsilon) = \varepsilon \sum_{i=0}^{\infty} \eta_i(\xi)\varepsilon^i, \tag{8}$$

$$\gamma(\xi, \varepsilon) = \varepsilon^2 \sum_{i=0}^{\infty} \gamma_i(\xi)\varepsilon^i. \tag{9}$$

We also allow the load speed to depend on $\varepsilon$ and write

$$v^2(\varepsilon) \equiv c(\varepsilon) = \sum_{i=0}^{\infty} c_i\varepsilon^i. \tag{10}$$

By symmetry considerations we infer that $\eta$ must be an odd function of $\varepsilon$ and $\gamma$ must be an even function of $\varepsilon$. So we set

$$\eta_i(\xi) \equiv 0, \qquad \gamma_i(\xi) \equiv 0, \qquad i \text{ odd}. \tag{11}$$

It then follows that only even powers of $\varepsilon$ need be retained in (10). So we also set

$$c_i \equiv 0, \qquad i \text{ odd}. \tag{12}$$

Before we proceed, it should be remarked that we assume in (10) $c_0 \neq 1$. It turns out that the two cases $c_0 < 1$ and $c_0 > 1$ must be investigated separately as follows.

*Case* 1: $c_0 < 1$

We substitute (8)–(12) into (1a) and (1b). Upon collecting like powers of $\varepsilon$ we obtain the following set of equations

$$\eta_0'''' + 2c_0\eta_0'' + \eta_0 = \delta(\xi), \tag{13}$$

$$(\lambda^2 - c_0)\gamma_0'' - \beta\gamma_0 = -\tfrac{1}{2}\lambda^2[(\eta_0')^2]', \tag{14}$$

$$\begin{aligned} \eta_2'''' + 2c_0\eta_2'' + \eta_2 = {} & -2c_2\eta_0'' + \lambda^2[2\eta_0''\gamma_0' + 2\eta_0'\gamma_0'' + 3(\eta_0')^2\eta_0''] \\ & + \alpha\eta_0^3 + 3\mu[2\eta_0''(\eta_0''')^2 + (\eta_0'')^2\eta_0''''], \end{aligned} \tag{15}$$

$$\cdots\cdot$$

To these equations we again supplement the conditions of continuity and of boundedness of the solutions.

Equation (13) has for $c_0 < 1$ the solution given in (3) which we rewrite as

$$\eta_0(\xi) = A_1 \operatorname{Re}\{\exp[i(|\xi|e^{i\Phi} - \Phi)]\}, \tag{16}$$

where now

$$A_1 = \tfrac{1}{2}(1-c_0^2)^{-\frac{1}{2}}, \qquad \Phi = \cos^{-1}\left(\frac{1+c_0}{2}\right)^{\frac{1}{2}}. \tag{17}$$

We assume further that the total transverse displacement under the load in the nonlinear response is set *a priori* and equal to $\varepsilon A_1 \cos\Phi$. This implies

$$\eta(0,\varepsilon) = \varepsilon A_1 \cos\Phi, \tag{18}$$

or equivalently,

$$\begin{aligned} \eta_0(0) &= A_1 \cos\Phi, \\ \eta_i(0) &= 0, \qquad i = 2, 4, \ldots \end{aligned} \tag{19}$$

$A_1$ will hereafter be referred to as the "amplitude parameter", or simply the "amplitude" of the nonlinear response, since in the range of load speeds considered here $\cos\Phi \simeq 1$. It follows from the definition of $A_1$ that

$$c_0 = \left(1-\frac{1}{4A_1^2}\right)^{\frac{1}{2}} (<1). \tag{20}$$

To solve equation (14) for $\gamma_0$ we may make use of the Green's function

$$G_1(\xi,\zeta) = -(\kappa/2\beta)\exp(-|\xi-\zeta|\kappa), \tag{21}$$

where

$$\kappa = [\beta/(\lambda^2-c_0)]^{\frac{1}{2}}, \qquad \beta \neq 0. \tag{22}$$

We then have

$$\gamma_0(\xi) = \int_{-\infty}^{\infty} [\text{inhomogeneous terms in (14)}].\, G_1(\xi,\zeta)\,\mathrm{d}\zeta$$

$$= \begin{cases} \dfrac{1}{2}A^2\lambda^2\,\mathrm{sgn}(\xi)\left\{\dfrac{\sin\Phi}{4(\lambda^2-c_0)\sin^2\Phi-\beta}[\exp(-2|\xi|\sin\Phi)-\exp(-\kappa|\xi|)]\right. \\ \qquad \left. +\,\mathrm{Im}\,\dfrac{e^{i\Phi}}{4(\lambda^2-c_0)\sin^2\Phi+\beta}[\exp(2i|\xi|e^{i\Phi})-\exp(-\kappa|\xi|)]\right\}, \\ \qquad \text{for } 4(\lambda^2-c_0)\sin^2\Phi-\beta \neq 0, \\ \dfrac{1}{2}A^2\lambda^2\left\{-\dfrac{\sin\Phi}{2\kappa(\lambda^2-c_0)}\xi\exp(-\kappa|\xi|)+\mathrm{Im}\,\dfrac{e^{i\Phi}}{4(\lambda^2-c_0)\sin^2\Phi+\beta}\right. \\ \qquad \left. .\,[\exp(2i|\xi|e^{i\Phi})-\exp(-\kappa|\xi|)]\,\mathrm{sgn}(\xi)\right\}, \\ \qquad \text{for } 4(\lambda^2-c_0)\sin^2\Phi-\beta = 0, \end{cases} \tag{23}$$

where the detailed calculations have been omitted.

The solution given in (16) yields the Green's function for equation (15)

$$G_2(\xi,\zeta) = \frac{1}{2(1-c_0^2)^{\frac{1}{2}}}\,\mathrm{Re}\{\exp[i(|\xi-\zeta|e^{i\Phi}-\Phi)]\}. \tag{24}$$

Consequently, we may represent the solution $\eta_2$ of equation (15) as

$$\eta_2(\xi) = \int_{-\infty}^{\infty} [\text{inhomogeneous terms in (15)}] \,.\, G_2(\xi, \zeta)\, \mathrm{d}\zeta. \tag{25}$$

We note that $c_2$ is so far undetermined in equation (15). To determine $c_2$ we use the condition $\eta_2(0) = 0$ as given in (19). Thus, from (25),

$$0 = \int_{-\infty}^{\infty} [\text{inhomogeneous terms in (15)}] \,.\, G_2(0, \zeta)\, \mathrm{d}\zeta, \tag{26}$$

and we find

$$c_2 = (\lambda^2 I_2 + \alpha I_3 + 3\mu I_4)/2I_1, \tag{27}$$

where

$$\left.\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \eta_0''(\zeta) G_2(0, \zeta)\, \mathrm{d}\zeta, \\
I_2 &= \int_{-\infty}^{\infty} [2\eta_0''(\zeta)\gamma_0'(\zeta) + 2\eta_0'(\zeta)\gamma_0''(\zeta) + 3(\eta_0'(\zeta))^2 \eta_0''(\zeta)] \,.\, G_2(0, \zeta)\, \mathrm{d}\zeta, \\
I_3 &= \int_{-\infty}^{\infty} \eta_0^3(\zeta) G_2(0, \zeta)\, \mathrm{d}\zeta, \\
I_4 &= \int_{-\infty}^{\infty} [2\eta_0''(\zeta)(\eta_0'''(\zeta))^2 + (\eta_0''(\zeta))^2 \eta_0''''(\zeta)] G_2(0, \zeta)\, \mathrm{d}\zeta.
\end{aligned}\right\} \tag{28}$$

It then follows that $\eta_2(\xi)$ is completely determined by (25). We shall not present the result for $\eta_2(\xi)$ explicitly. We terminate our calculations at this order and shall return to examine these results after the case $c_0 > 1$ is treated.

*Case* 2: $c_0 > 1$

If we proceed here for $c_0 > 1$ in the same way as in the previous case, we find that secular terms appear in the equation for $\eta_2(\xi)$ which render the solution unbounded as $\xi$ tends to $\pm\infty$. To remedy this situation we introduce another parameter $\Omega$, which will be identified later as the square of the spatial wave number of the solution $\eta$, to depend on $\varepsilon$. We write

$$\Omega = \sum_{i=0}^{\infty} \Omega_i \varepsilon^i, \tag{29}$$

with

$$\Omega_i = 0, i \text{ odd}. \tag{30}$$

We then substitute (8)–(12) as well as (29) and (30) into (1a) and (1b). At this stage we introduce a new independent variable $s$ related to $\xi$ by

$$s = \Omega^{\frac{1}{2}} \xi. \tag{31}$$

Upon the substitutions and collecting like powers of $\varepsilon$ as before, we obtain

$$\Omega_0^2\frac{d^4\eta_0}{ds^4}+2c_0\Omega_0\frac{d^2\eta_0}{ds^2}+\eta_0 = \Omega_0^{\frac{1}{2}}\delta(s), \tag{32}$$

$$\Omega_0(\lambda^2-c_0)\frac{d^2\gamma_0}{ds^2}-\beta\gamma_0 = -\lambda^2\Omega_0^{\frac{3}{2}}\frac{d\eta_0}{ds}\frac{d^2\eta_0}{ds^2}, \tag{33}$$

$$\begin{aligned}\Omega_0^2\frac{d^4\eta_2}{ds^4}+2c_0\Omega_0\frac{d^2\eta_2}{ds^2}+\eta_2 &= -2\Omega_0\Omega_2\frac{d^4\eta_0}{ds^4}-2(c_2\Omega_0+c_0\Omega_2)\frac{d^2\eta_0}{ds^2}\\ &+3\lambda^2\Omega_0^2\left(\frac{d\eta_0}{ds}\right)^2\frac{d^2\eta_0}{ds^2}+2\lambda^2\Omega_0^{\frac{3}{2}}\frac{d}{ds}\left(\frac{d\eta_0}{ds}\cdot\frac{d\gamma_0}{ds}\right)+\alpha\eta_0^3\\ &+3\mu\Omega_0^4\left[2\frac{d^2\eta_0}{ds^2}\left(\frac{d^3\eta_0}{ds^3}\right)^2+\left(\frac{d^2\eta_0}{ds^2}\right)^2\frac{d^4\eta_0}{ds^4}\right]+\tfrac{1}{2}\Omega_0^{-\frac{1}{2}}\Omega_2\delta(s),\\ &\dots\end{aligned} \tag{34}$$

Equation (32) is recognized as the linear beam equation (2), and hence for $c_0 > 1$ has the solution given in (5) which we rewrite as

$$\eta_0(s) = -\frac{A_2}{\Omega_a^{\frac{1}{2}}}\sin\left(\frac{\Omega_a}{\Omega_0}\right)^{\frac{1}{2}}s\,.\,H(-s)-\frac{A_2}{\Omega_b^{\frac{1}{2}}}\sin\left(\frac{\Omega_b}{\Omega_0}\right)^{\frac{1}{2}}s\,.\,H(s), \tag{35}$$

with $\Omega_{a,b}$ being defined in (6) and the "amplitude parameter" $A_2$ is related to $c_0$ by

$$A_2 = \tfrac{1}{2}(c_0^2-1)^{-\frac{1}{2}}, \qquad c_0 = \left(1+\frac{1}{4A_2^2}\right)^{\frac{1}{2}}. \tag{36}$$

The solution of equation (33) for $\gamma_0$ can now be determined, say by using the Green's function $G_1(\xi,\zeta)$ as given in (21). It is found as

$$\begin{aligned}\gamma_0(s) &= \frac{1}{2}A_2^2\lambda^2\left[-\frac{\Omega_a^{\frac{1}{2}}}{4\Omega_a^{\frac{1}{2}}(\lambda^2-c_0)+\beta}\sin 2\left(\frac{\Omega_a}{\Omega_0}\right)^{\frac{1}{2}}s+B\exp(\kappa s/\Omega_0^{\frac{1}{2}})\right]H(-s)\\ &+\frac{1}{2}A_2^2\lambda^2\left[-\frac{\Omega_b^{\frac{1}{2}}}{4\Omega_b^{\frac{1}{2}}(\lambda^2-c_0)+\beta}\sin 2\left(\frac{\Omega_b}{\Omega_0}\right)^{\frac{1}{2}}s+B\exp(-\kappa s/\Omega_0^{\frac{1}{2}})\right]H(s),\end{aligned} \tag{37}$$

where

$$B = \frac{1}{\kappa\Omega_0}\cdot\frac{\beta(\Omega_a-\Omega_b)}{16(\lambda^2-c_0)^2+8c_0\beta(\lambda^2-c_0)+\beta^2}. \tag{38}$$

The solutions for $\eta_0$ and $\gamma_0$ as given in (35) and (37) are now substituted into equation (34). So that the equation may have a bounded solution for $\eta_2$ as $s$ tends to $\pm\infty$, secular terms appearing on the right hand side of equation (34) must be removed. This means we must set the coefficients of the terms containing $\sin(\Omega_a/\Omega_0)^{\frac{1}{2}}s$ and $\sin(\Omega_b/\Omega_0)^{\frac{1}{2}}s$ respectively equal to zero. By the assumption $\lambda^2 \gg c_0, \beta$, this leads to the following equations

$$c_2 + \frac{\Omega_2}{\Omega_0}(c_0 - \Omega_a) \cong \frac{A_2^2}{8}\left(2\lambda^2 - \frac{3\alpha}{\Omega_a^2} - 3\mu\Omega_a^2\right), \tag{39a}$$

$$c_2 + \frac{\Omega_2}{\Omega_0}(c_0 - \Omega_b) \cong \frac{A_2^2}{8}\left(2\lambda^2 - \frac{3\alpha}{\Omega_b^2} - 3\mu\Omega_b^2\right), \tag{39b}$$

from which $c_2$ and $\Omega_2$ are determined. Equation (34) then determines $\eta_2(\xi)$. Again we shall not present $\eta_2(\xi)$ here explicitly and shall terminate our calculations at this order.

To summarize our results up to this point: we have obtained two sets of perturbation solutions, distinguished now by the superscripts 1 and 2, in the form

$$\eta^{(i)}(\xi, \varepsilon) = \varepsilon\eta_0^{(i)}(\xi) + \varepsilon^3\eta_2^{(i)}(\xi) + O(\varepsilon^5), \qquad i = 1, 2, \tag{40}$$

$$\gamma^{(i)}(\xi, \varepsilon) = \varepsilon^2\gamma_0^{(i)}(\xi) + O(\varepsilon^4), \qquad i = 1, 2, \tag{41}$$

and

$$c^{(i)}(\varepsilon) = c_0^{(i)} + \varepsilon^2 c_2^{(i)} + O(\varepsilon^4), \qquad i = 1, 2, \tag{42}$$

with $c_0^{(1)} < 1$ and $c_0^{(2)} > 1$ respectively. The perturbation parameter $\varepsilon$ is the load intensity $\Delta$. For a fixed $\Delta$ these solutions are completely determined by specifying the "amplitudes" $A_1$ and $A_2$ which are related to $c_0^{(i)}$ through (20) and (36) and hence to $v^2 = c^{(i)}$ through (42). The significance of these perturbation solutions is that depending on the algebraic signs of $c_2^{(i)}$, the actual (total) speeds may in fact have covered the critical speed even if the unperturbed speeds are strictly subcritical ($c_0 < 1$) or supercritical ($c_0 > 1$). These solutions will be examined in more detail below.

## RESULTS AND DISCUSSIONS

For $c_0$ close to unity we have $\sin\Phi = [(1-c_0)/2]^{\frac{1}{2}} \cong 0$ and $\cos\Phi \cong 1$ [see equation (17)]. We then obtain from equation (27)

$$c_2^{(1)} \cong \frac{A_1^2}{16}(2\lambda^2 - 3\alpha - 3\mu) \quad \text{for } c_0 < 1. \tag{43}$$

As equations (39) yield

$$c_2^{(2)} \cong \frac{A_2^2}{8}(2\lambda^2 - 3\alpha - 3\mu) \quad \text{for } c_0 > 1, \tag{44}$$

it is seen that $c_2^{(i)}$ has the same algebraic sign as the quantity $(2\lambda^2 - 3\alpha - 3\mu)$. The constant $\lambda$ characterizes the geometrical nonlinearity in the beam while $\mu$ and $\alpha$ characterize the nonlinearity of the elastic beam and foundation respectively ($\mu, \alpha > 0$ for soft materials). Thus $c_2^{(i)} > 0$ when the geometrical nonlinearity is predominant and $c_2^{(i)} < 0$ when the material nonlinearity is predominant. The criterion is identical with the one obtained by Steele [2].

When $c_2^{(1)} > 0$, the "subcritical" solutions (with the superscript "1") are valid up to the critical speed and even beyond. The same is true with the "supercritical" solutions (with the superscript "2") when $c_2^{(2)} < 0$. In order to see this we consider a numerical example. Let us take the following parameter values:

$$\lambda^2 = 25,$$
$$\alpha = 0, 50,$$
$$\mu = 0,$$
$$\beta = 1{\cdot}0,$$
$$\Delta = 0{\cdot}005, 0{\cdot}010, 0{\cdot}015.$$

In Figs. 1 and 2 the solid curves show the dependences of the maximum amplitudes of the nonlinear responses upon the load speed $v = \sqrt{c}$. The broken curves in the same figures show the corresponding relationships according to the linear beam theory. We recall that for a subcritical mode of response ($c_0 < 1$) the maximum amplitude occurs under the load and is given by $\Delta A_1 \cos \Phi$. For a supercritical mode of response ($c_0 > 1$) the maximum amplitude occurs behind the load and is given by $\Delta A_2/\Omega_a^{\frac{1}{4}}$ when the contribution of $\eta_2^{(2)}(\xi)$ is neglected [see equation (35)]. Figure 1 corresponds to a case where the geometrical non-linearity in the beam is predominant, and it shows that the curves associated with the sub-

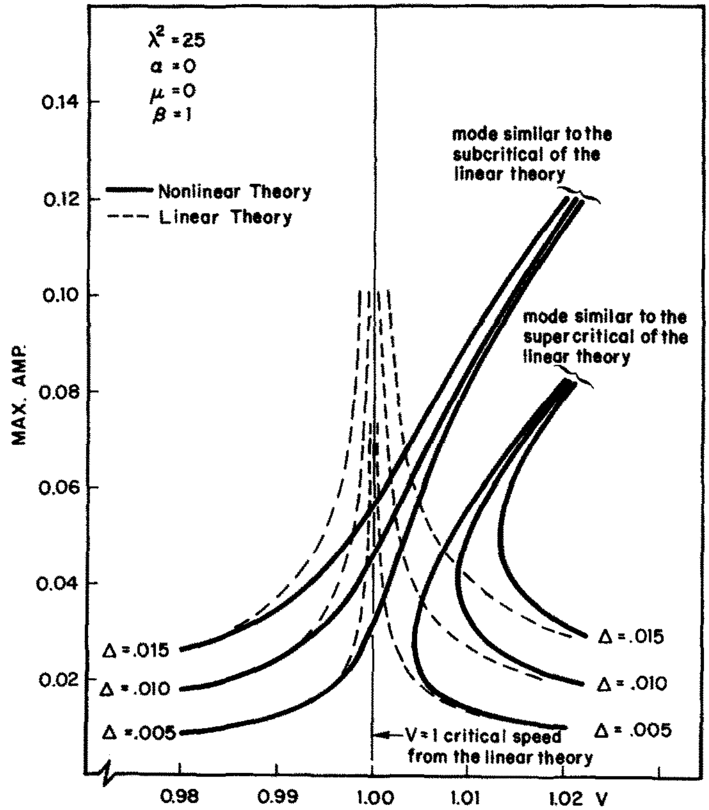


FIG. 1. Maximum amptitudes of beam responses vs. load speeds.

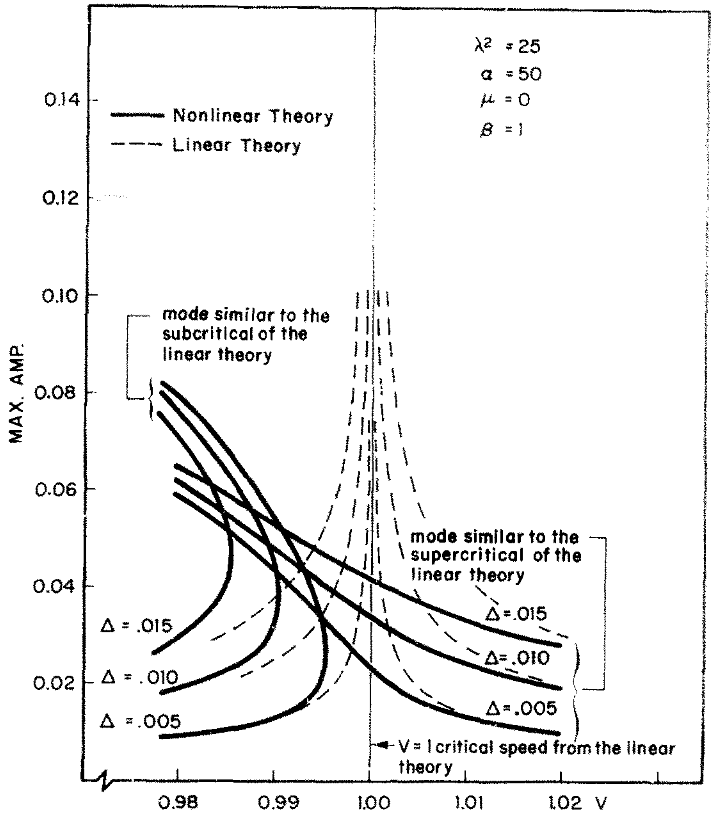


FIG. 2. Maximum amplitudes of beam responses vs. load speeds.

critical mode of responses are extended into the region $v > 1$, while the curves associated with the supercritical mode of responses do not continue into the region $v < 1$. Figure 2 corresponds to a case where the material nonlinearity is predominant and in it the above situation is just reversed. These curves resemble those "response curves" in nonlinear vibration studies involving soft- and hard-spring characteristics.

These curves show that for a fixed load intensity, the transition from a subcritical mode of response to a supercritical mode of response as the load speed is increased, and vice versa as the load speed is decreased, takes place at a speed slightly greater than the critical speed if the geometrical nonlinearity in the beam is predominant. On the other hand, if the material nonlinearity is predominant, such a transition will then take place at a speed slightly smaller than the critical speed. Furthermore, a particular mode of response may have two different amplitudes at a load speed near the critical speed (when the curves bend back). We may conjecture that the response corresponding to the smaller amplitude is the stable one.

In conclusion we mention that the two perturbation solutions obtained by Steele [2] valid for the critical speed only may be shown to agree with the present solutions. The subcritical and supercritical solutions obtained by Steele, however, can not be continued to the critical speed because the leading terms in such solutions, i.e. the respective linear solutions, become unbounded. The present solutions, obtained by using the special perturbation method, show how Steele's solutions are related to one another.

## REFERENCES

[1] J. T. KENNEY, JR., Steady-state vibrations of beam on elastic foundation for moving loads. *J. Appl. Mech.* **21**, 359 (1954).
[2] C. R. STEELE, Nonlinear effects in the problem of the beam on a foundation with a moving load. *Int. J. Solids Struct.* **3**, 565 (1967).
[3] D. H. Y. YEN and S. C. TANG, On the nonlinear response of an elastic string to a moving load. *Int. J. nonlinear Mech.* **5**, 465 (1970).

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**Абстракт**—Исследуется нелинейное воздействие упругой балки, подверженной действию движущейся поперечной нагрузки, при использовании специального метода возмущения. Получаются решения, которые остаются важными в каждой точке некоторой области критической скорости для линейной теории балок. Находится, что вообще, в зависимости от типа преобладающей нелинейности в болке, так подкритическое как и сверхкритическое воздействие могут быть постоянными вплоть критической скорости и даже выше. Решения указывают также, что переходы от подкритихеского воздействия к сверкритическому и наоборот могут существовать близи критической скорости.